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VOL. 10 BATON ROUGE, LA., APRIL, 1936 No. 7

The Gifted Student

Entrance Requirements and Their Effect Upon Mathematics In the High School

On a Certain Integral Arising In Quantum Mechanics

An Approximate Formula for Evaluating Factorial Numbers

The Teacher's Department

Mathematical Notes

Problem Department

Book Reviews

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1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Gifted Student

More than a decade ago a survey known as "The Gifted Student Project" was made under the auspices of the National Research Council, by Dean Seashore of the University of Iowa. In his report was expressed the motto "Keep each student busy at his highest natural level of successful achievement in order that he may be happy, useful and good." However students may have been created equal, their intellects are varied. To carry out the motto in larger institutions, the practice of sectionizing students on the basis of their abilities is adopted. This is proper in mathematics where superiority is evidenced by interest and the ability to generalize and to abstract—an ability markedly deficient in the average student.

A mixture of abilities in the same class leads to the humiliation of the dull student, to a habit of failure and to final hopelessness. It retards the brilliant student and may result in his mental stagnation. When the National Research Council made its report, some form of sectionizing on the basis of ability was being used in approximately half of the leading institutions of the United States.

For the past two years, we have put our superior students into "Honor Sections," membership being determined on the basis of past records and native intelligence. Our conduct of an Honor Section embraces little of the lecture method. Where lecturing is in order, it is made as concise as possible. The students are repeatedly reminded to so focus their minds on the logical processes involved that no vagueness shall remain in their thought about what was done and why. Concentration and attention are thus strengthened by the conscious effort. According to psychology, things learned when the mind is geared to high pitch are clearer and remain longer in the consciousness. The objective in our program is not breadth but depth. Investigation is fostered by proposing problems of difficulty for those with the time and with access to books. Initiative, resourcefulness and independence of thought have been increased by stimulating the superior student with the desire to know and to think with effectiveness.

R. L. O'QUINN,
Louisiana State University.

*Entrance Requirements and Their Effect Upon Mathematics in the High School

By E. R. SLEIGHT
Albion College

For a number of years many of the Superintendents and Principals of the high schools in Michigan were dissatisfied with the fact that some of their most promising graduates could not qualify for entrance into college. This condition was brought about by the fact that practically all of the colleges belonging to the North Central Association demanded the same number and type of units for entrance. Of the total of 15 units required, only three could come from the so-called vocational subjects. The remaining 12 units included at least three of English, two in mathematics, one in a laboratory science and two in the same foreign language. It was felt among the high school administrators that more units of credit should be allowed for the vocational subjects. In order that this matter might be thoroughly investigated, the Michigan †Educational Association appointed a committee from its members. To this committee were added several others, representing the various types of collegiate institutions in the state. It happened to be my privilege to represent the denominational colleges.

For the first few meetings of this group the discussion centered around the vocational subjects, and how to formulate a plan that would allow more lee-way from this particular angle. It was finally decided to reduce the number of so-called solids from 12 units to 10, permitting the other five units to be selected from any additional subjects that the high school allowed to count toward graduation. But the basis on which to select the 10 units was a matter of many heated discussions. Finally the sequence idea was decided upon, and the committee began building around this plan. From the subjects offered in a high school, five groups were chosen,—English, Foreign Languages, Social Sciences, Natural and Physical Sciences, and Mathematics. To be a candidate for college a graduate from a high school should present two major sequences and two minor sequences,

*Committee appointed 1929.

†Read at Mathematics Section Michigan Education Association, Jackson, Michigan October, 1935.

a major sequence being defined as three units from any one of the five groups, while a minor sequence should contain only two units. It was proposed that a major sequence from the English group be required of all students entering college, thus leaving four groups from which to select the remaining three sequences.

The committee presented its plan to the colleges of the state for their endorsement, but it was not accepted immediately. In fact a period of years passed before a similar plan was adopted and set into operation, beginning with the present college year. The fact that several other institutions are operating on a plan similar to the one above described had something to do with the final decision of the Michigan colleges. The University of Illinois has a plan almost identical, the only exception being that three majors are accepted as fulfilling entrance requirements, as well as two majors and two minors. Leland Standford, University of Chicago, Northwestern University, and the University of Oregon are among the larger institutions no longer requiring mathematics for entrance.

Since there are five groups and only four required sequences, it is possible for a student to enter college with no training in mathematics. Just what effect these changed entrance requirements will have upon mathematics in the high school can be determined only after a period of operation. Judging from the past experience it is likely that considerable change will result not only because of the entrance requirements themselves, but also because of the effect they will have upon the attitude of the state toward the subject of mathematics, caused by the fact that a high school graduate no longer requires mathematics to enter college.

When I began to study this question, my first reaction was one of decided concern. But more and more I am becoming convinced that mathematics will still hold a prominent place in the scheme of high school education.

In order to get something of a background for my thinking I wrote letters to several of the principals and superintendents in the state asking them to give me their reaction on this question of entrance requirements and mathematics in the high school. The majority of the replies indicate that this subject will continue to be required, or else elective, as in the past. Quoting from some of these replies we have the following pertinent opinions:

1. "Very little difference in this high school, as we do not contemplate any change in our policy of requiring one and one-half units of algebra and one unit of geometry for graduation." This was the

reply of a principal of one of the Detroit schools, and he seems to think that he is voicing the opinion of the Detroit schools in general.

2. A statement from the superintendent of one of the best high schools in the northern part of the state seems to voice the sentiment of many of the replies. "It is my opinion that there will be a decrease in enrollments in mathematics, but this will have a tendency to raise the level of the class, thus making it possible to do more and better work."

3. From the capitol city comes this statement: "After talking this matter over with the high school principals we have come to the conclusion that there will be very little decrease in the numbers taking mathematics. If a student wishes to go to college he is almost forced to decide between mathematics and a language, and it is our opinion that the decision will be in favor of the former. Also, mathematics has been required so long, and its uses are so varied, that people have come to think of it as necessary."

4. All replies are not so encouraging as the following indicates: "I believe that there will be a great decrease in the number of students electing algebra and geometry. Within two years only those students will elect these subjects who are definitely interested."

But the college entrance requirements are not the only requirements that have had a tendency to insure mathematics a place in the high school curriculum. In the majority of the states there has been a state ruling which compelled the schools to require a certain amount of training in mathematics for graduation from an approved high school. *The National Mathematics Magazine* has gathered some very interesting information bearing on this particular phase of the question. The following* questions were addressed by the Editor of this magazine to the State Superintendents of Education of the 48 states in the United States:

1. Has mathematics as a required subject for graduation from high school been eliminated in your state?

2. If it has not been eliminated is there a definite prospect that it will be at an early date?

3. In your judgment are there good reasons for the hope that mathematics will have increased rather than diminished use in the secondary schools of your state?

Answers to the first question brought out the following facts: In 32 of the states there is still a state requirement in mathematics, the amount and content varying greatly. In some states this require-

*Mathematics News Letter, May, 1934.

ment is satisfied by one year of general mathematics, while in others a boy or girl cannot graduate from high school without one and one-half units of algebra together with plane and solid geometry. In 14 of the states there is no state requirement, but the majority of the schools are controlled by local regulations requiring a certain amount of mathematics. Six states indicate a definite antagonism when it comes to any sort of fixed requirement in this field. This list of six is rather significant, as it includes some states which have made outstanding contributions to educational problems.

With regard to the second of these questions, of the 32 states now requiring some training in mathematics for graduation from high school, 23 indicated a very decided tendency to continue on the present plan, while four are just as decidedly in favor of abolishing any sort of requirement. The remaining five all suggest that a change in content in traditional courses is necessary if the requirement is to be retained.

The third question brought out no great response. In the very nature of the question, an answer can be only a matter of opinion,—and one man's opinion is as good as another, especially as all parties involved had the same background of information.

Just what effect, then, will all of this tendency toward decreased requirements of the amount of mathematics have upon the subject in the high schools of the future? At present about 75 per cent of the high schools in Michigan are classed as small high schools, and their program of study is very generally directed by the State Department of Public Instruction. In such cases mathematics has been required for graduation. How much of such a requirement has been determined by the college entrance requirements is an unknown factor, but it is my opinion that these entrance requirements have very materially determined the type of program set up by the state. With the changed condition, it is very likely that some of the required subjects will become elective, and mathematics will be no exception. Two years ago the University of Illinois published its new sequence plan for entrance. In reply to the first question sent out by the *National Mathematics Magazine*, Harry M. Thrasher, State High School Inspector for Illinois makes this statement: "Mathematics has never been required by this office for graduation from a recognized high school. Most schools, however, have been guided by the five fixed units set up by the various colleges and universities in the state. These are as follows: English three, algebra one, and geometry one. Last year, however, the University of Illinois put its entrance requirements on a different basis, and has eliminated the fixed subjects. Under these new requirements a candidate seeking entrance to the

University must present at least three majors, or two majors and two minors. It is possible for him to secure these requirements without having had any mathematics." This clearly implies that, in the opinion of the State Inspector, mathematics is not likely to be stressed in the high school as in the past. Many of the answers to this first question point out that in many schools the lack of electives will make it necessary to include mathematics, and I predict that this will be found true in many of the smaller schools in Michigan. But in those schools in which there is an opportunity to offer a sufficiently varied program, I confidently expect to see fewer enrollments in algebra and geometry.

From the replies to the question which I sent out to the Superintendents and Principals of the state, as well as from the results of the investigation made by the *National Mathematics Magazine* we are in a position to draw some definite conclusions. Whether these conclusions are correct or not time alone can tell.

In the first place mathematics will be in competition with other subjects. It seems to me that under the new plan, we can look for an increased demand for the social sciences. I predict that the majority of students entering college will present a major in this field in addition to the required major in English. This then leaves three fields for the two minors. Since mathematics aids in a thorough understanding of the sciences, and since students interested in the sciences are usually interested in the mathematics, it would seem that these two fields will naturally be represented in many cases. Also I believe that a student electing a language is just as likely to elect the mathematics as the sciences,—and perhaps a bit more so. Therefore from the standpoint of competition, it would seem that mathematics should come through in fairly good shape. But we must look for some decrease from this source.

Then there is the question of the usefulness of the subject. When we think of its application to our social, economic, and industrial life, as well as to the fields of science and engineering we are led to believe that high school pupils should be urged to take at least elementary algebra and plane geometry. If a college student decides that he wishes to enter some field in which mathematics is required, and he has had no training along this line, he will be greatly handicapped, and possibly will even be prevented from doing the thing he wishes to do. Mathematics, like music, cannot be learned over night.

But with all the elements favoring elections in mathematics, I am confident that there will be a general falling off in enrollments in our high school classes. A certain type of student ought not to be

compelled to take algebra or geometry. I am sure that every teacher of these subjects will appreciate their absence. This group will be no loss what-so-ever. It will make it possible to do much better work, and to cover more material, as these are the ones who retard the progress of the class, and sap the vitality of the teacher.

The extent to which a further decrease will occur will depend in some measure upon the content of the course and the manner in which it is presented. As indicated previously several of the states are investigating this problem. But in the last analysis mathematics will continue to be in great demand only as the teachers are in a position to make the subject both interesting and useful. She should be well trained to teach every bit of the work required, and should know how to gear the subject matter into life as it exists today. Mathematics is demanded more and more, even in fields hitherto untouched, and I believe we can double the interest in the thing we teach if we can show how necessary it is to human development. I believe I do the subject of Latin no great harm when I state that in most schools it is not the most popular subject in the entire array of offerings in our high schools. But here and there we find a teacher who has made it so interesting that her classes are filled to capacity. If we compare the applications of Latin with the number of ways in which mathematics may be used, we have a running start on making it a live interesting subject to the high school boy and girl. The extent to which mathematics will be in demand in many schools, so far as the future is concerned, will depend very materially upon those of us who prepare the teachers, and upon those who teach. It is a good thing to have something to spur us on, if we will only accept the challenge.

On a Certain Integral Arising In Quantum Mechanics

By H. E. BUCHANAN
Tulane University

1. In the consideration of a research problem in quantum mechanics, Professor J. C. Morris of Princeton University recently encountered the integral

$$I = \int_0^\infty \frac{x^3\,dx}{e^x - 1}. \tag{1}$$

Since the integral does not yield to any ordinary methods of attack, Professor Morris asked the author to evaluate it. The evaluation affords an interesting exercise in several phases of advanced calculus, and thus may be of interest to students and teachers of senior or first year graduate grade.

2. *Existence.* At the lower limit, there is no difficulty, for although the function under the integral sign becomes indeterminate, the indeterminate form is easily evaluated, thus:

$$\lim_{x=0} \frac{x^3}{e^x - 1} = \lim_{x=0} \frac{3x^2}{e^x} = 0.$$

At the upper limit the function is in the indeterminate form ∞/∞ but evaluation shows that it approaches zero as x becomes infinite.

Since the upper limit is infinite the integral has a finite value* if there exists a $\mu > 1$ such that

$$\lim_{x \to \infty} \frac{x^{\mu+3}}{e^x - 1}$$

has a finite value. Differentiation shows that for every value of $\mu > 1$, this limit is zero. Hence the integral exists.

*Goursat-Hedrick, *Mathematical Analysis*, vol. I, p. 181, First Edition.

3. *Evaluation.* It remains to find the value of the integral. In (1) let

$$x = \log 1/z = -\log z \; ; \; dx = -dz/z \; ; \; e^x - 1 = 1/z - 1$$

Hence
$$I = -\int_0^1 \frac{(\log z)^3 \, dz}{1-z}.$$

Expanding $1/(1-z)$ we have

$$I = -\int_0^1 (1+z+z^2+z^3+\ldots.+z^m+\ldots.)(\log z)^3 \, dz. \quad (2)$$

Integration by parts easily gives

$$\int_0^1 x^{\alpha-1} \log x \, dx = -\frac{1}{\alpha^2} \qquad (\alpha = 1,2,\ldots) \quad (3)$$

Differentiating both members of (3) with respect to α twice,

$$\int_0^1 x^{\alpha-1} (\log x)^3 \, dx = -\frac{6}{\alpha^4} \quad (4)$$

Now if we integrate (2) term by term, each integral is of the type (4). It follows that

$$\int_0^\infty \frac{x^3 \, dx}{e^x - 1} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots.\right) = 6.49\ldots.$$

It had been found from other considerations that the integral should have a value between 6.3 and 6.9. Thus the value above furnishes a theoretical verification of experimental results.

The series $\sum_{n=1}^{\infty} 1/n^4$ may be given in closed form.* Its sum is $\Pi^4/90$ so that the value of the integral is $\Pi^4/15$.

*[Knopp, Theorie und Anwendung der Unendlichen Reihen, Zweite Auflage, p. 239]

A more general problem is solved by Titchmarsh on page 45 of his *Theory of Functions* and the same problem is covered indirectly by Whittaker and Watson.

The method of deriving new integrals by differentiation under the integral sign as well as the conditions under which such differentiation is allowable are discussed in Goursat-Hedrick, *Mathematical Analysis*, vol. I, first edition, page 192, and W. F. Osgood has an interesting article on the same subject on page 119, *Annals of Mathematics*, 1908.

On N+2 Mutually Orthogonal Hyperspheres

By FRANK AYRES, Jr.
Dickinson College

I. Introduction. Let an arbitrary point in Euclidean n−space be designated by

$$(x) \equiv (x^{(1)}, x^{(2)}, \ldots\ldots, x^{(n)}).$$

Then, if $P_i \equiv (x_i)$, $(i=1,2,\ldots,n+2)$, are the centers of $n+2$ mutually orthogonal hyperspheres S_i of radius r_i, we have

$$1) \qquad \sum(x_i - x_j)^2 = r_i^2 + r_j^2, \ (i,j=1,2,\ldots,n+2;\ i \neq j)$$

which, if we write

$$2) \qquad \sum x_k^2 = r_k^2 + 2c_k$$

takes the form

$$3) \qquad \sum x_i x_j = c_i + c_j.$$

The points P_i constitute an orthocentric system in the space.*

Let T_i be the $(n+1)$−point whose vertices are $P_1, P_2, \ldots P_{i-1}, P_{i+1}, \ldots, P_{n+2}$. The square of its measure, V_i, is given by

$$4) \qquad (n!)^2 V_i^2 = \prod_{j=1}^{n+1} r_{i+j}^2 \sum_{j=1}^{n+1} 1/r_{i+j}^2 = -1/r_i^2 \prod_{j=1}^{n+1} r_{i+j}^2 \equiv C_i^2$$

(with the convention, which will hold throughout this paper, that any subscript $i+j$ is to be reduced modulo $n+2$ if $i+j > n+2$), from which follows

$$5) \qquad \sum_{j=1}^{n+2} 1/r_j^2 = 0.$$

The coordinates (x_{jk}) of the point of intersection P_{jk} of the edge P_jP_k of T_i and the hyperplane determined by the remaining vertices and P_i are given by

$$(x_{jk}) = \left(\frac{r_j^2 x_k + r_k^2 x_j}{r_j^2 + r_k^2} \right).$$

*The results given in this section are found in a paper: G. E. Raynor: On N+2 Mutually Orthogonal Hyperspheres in Euclidean N−space, *American Mathematical Monthly*, vol. 41 (1934), p. 424.

2. Radius of the hypersphere circumscribing an $(n+1)-$point. We take the center O_i of the hypersphere of radius R_i circumscribing the $(n+1)-$point T_i as the origin. Then, we may write

$$C_i^2 \equiv \begin{vmatrix} x_{i+1}^{(1)} & x_{i+1}^{(2)} \cdots x_{i+1}^{(n)} & 1 \\ x_{i+2}^{(1)} & x_{i+2}^{(2)} \cdots x_{i+2}^{(n)} & 1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{i+n+1}^{(1)} & x_{i+n+1}^{(2)} \cdots x_{i+n+1}^{2} & 1 \end{vmatrix}^2$$

$$= \begin{vmatrix} R_i^2+1 & R_i^2+1-\frac{1}{2}(r_{i+1}^2+r_{i+2}^2) & \cdots R_i^2+1-\frac{1}{2}(r_{i+1}^2+r_{i+n+1}^2) \\ R_i^2+1-\frac{1}{2}(r_{i+1}^2+r_{i+2}^2) & R_i^2+1 & \cdots R_i^2+1-\frac{1}{2}(r_{i+2}^2+r_{i+n+1}^2) \\ R_i^2+1-\frac{1}{2}(r_{i+1}^2+r_{i+3}^2) & R_i^2+1-\frac{1}{2}(r_{i+2}^2+r_{i+3}^2) & R_i^2+1-\frac{1}{2}(r_{i+3}^2+r_{i+n+1}^2) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ R_i^2+1-\frac{1}{2}(r_{i+1}^2+r_{i+n+1}^2) & R_i^2+1-\frac{1}{2}(r_{i+2}^2+r_{i+n+1}^2) & \cdots R_i^2+1 \end{vmatrix}$$

since, by 1),

$$\sum x_j x_k = R_i^2 - \tfrac{1}{2}(r_j^2+r_k^2), \quad (j,k \neq i).$$

In the above determinant, subtract the first row from the other rows and subtract the first column from the other columns. Next, multiply each row except the first by $\frac{1}{2}$ and add it to the first row; then, multiply each column except the first by $\frac{1}{2}$ and add it to the first column; finally, subtract $(n-1)/2$ times the last column from the first column and subtract the last column from the 2nd, 3rd,..., nth column. We obtain.

$$C_i^2 = \begin{vmatrix} \xi_i & 0 & 0 & \cdots & 0 & \frac{n-1}{2} r_{i+1}^2 \\ 0 & r_{i+2}^2 & 0 & \cdots & 0 & r_{i+1}^2 \\ 0 & 0 & r_{i+3}^2 & \cdots & 0 & r_{i+1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & r_{i+n}^2 & r_{i+1}^2 \\ -\frac{n-1}{2} r_{i+n+1}^2 & -r_{i+n+1}^2 & -r_{i+n+1}^2 & \cdots & -r_{i+n+1}^2 & r_{i+1}+r_{i+n+1}^2 \end{vmatrix}$$

$$= \xi_i r_{i+1}^2 r_{i+n+1}^2 \begin{vmatrix} r_{i+2}^2 & 0 & 0 & \dots & 0 & 1 \\ 0 & r_{i+3}^2 & 0 & \dots & 0 & 1 \\ 0 & 0 & r_{i+4}^2 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & r_{i+n}^2 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{vmatrix}$$

$$+ \xi_i \{ ((r_{i+1}^2)) + ((r_{i+n+1}^2)) \} + \frac{(n-1)^2}{4} \prod_{j=1}^{n+1} r_{i+j}^2,$$

where

$$\xi_i \equiv R_i^2 + 1 - \tfrac{1}{2} \sum_{j=i}^{n+1} r_{i+1}^2 \quad \text{and} \quad ((r_{i+k}^2)) \equiv \prod_{j=1}^{n+1} r_{i+j}^2 / r_{i+k}^2.$$

If in the above determinant, we subtract the $(n-1)$st column from the 1st, 2nd,...,$(n-2)$nd columns and expand along the last row, the result may be written in the form

$$r_{i+n}^2 \begin{vmatrix} r_{i+2}^2 & 0 & 0 & \dots & 0 & 1 \\ 0 & r_{i+3}^2 & 0 & \dots & 0 & 1 \\ 0 & 0 & r_{i+4}^2 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & r_{i+n-1}^2 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} r_{i+2}^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & r_{i+3}^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & r_{i+4}^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & r_{i+n-1}^2 & 0 \\ -r_{i+n}^2 & -r_{i+n}^2 & -r_{i+n}^2 & \dots & -r_{i+n}^2 & 1 \end{vmatrix}$$

and hence,

$$C_i^2 = \xi_i (r_{i+1}^2 r_{i+n}^2 r_{i+n+1}^2) \begin{vmatrix} r_{i+2}^2 & 0 & 0 & \dots & 0 & 1 \\ 0 & r_{i+3}^2 & 0 & \dots & 0 & 1 \\ 0 & 0 & r_{i+4}^2 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & r_{i+n-1}^2 & 1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{vmatrix}$$

$$+\xi_i\{((r_{i+1}^2))+((r_{i+n}^2))+((r_{i+n+1}^2))\}+\frac{(n-1)^2}{4}\prod_{j+1}^{n+1} r_{i+j}^2.$$

Repeating the above procedure, we obtain finally,

$$C_i^2=(R_i^2+1-\tfrac{1}{4}\sum_{j=1}^{n+1} r_{i+j}^2)\prod_{j=1}^{n+1} r_{i+j}^2\sum_{j=1}^{n+1} 1/r_{i+j}^2+\frac{(n-1)^2}{4}\prod_{j=1}^{n+1} r_{i+j}^2$$

whence, using 4) and 5), we have

$$6)\qquad R_i^2=\frac{(n-1)^2}{4}r_i^2+\tfrac{1}{4}\sum_{j=1}^{n+1} r_{i+j}^2$$

Following Court,† we shall call this hypersphere the associated central hypersphere of P_i.

3. Calculation of the c's. When O_i is taken as the origin, we have, by 2),

$$7)\qquad 2C_k=R_i^2-r_k^2,\quad (k\neq i)$$

In order to calculate the value of c_i in terms of R_i, we consider the set of relations

$$\sum x_i x_{i+j}=c_i+c_j\quad,\ (j=1,2,\ldots,n+1)$$

as $n+1$ equations in the $n+1$ unknowns

$$x_i^{(1)},\ x_i^{(2)},\ldots,\ x_i^{(n)},\ c_i.$$

We write

$$C_i=\frac{A_i}{B_i}=\frac{A_iB_i}{B_i^2}$$

where

$$B_i=\begin{vmatrix} x_{i+1}^{(1)} & x_{i+1}^{(2)} & \ldots. & x_{i+1}^{(n)} & -1\\ x_{i+1}^{(1)} & x_{i+2}^{(2)} & \ldots. & x_{i+2}^{(n)} & -1\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ x_{i+n+1}^{(1)} & x_{i+n+1}^{(2)} & \ldots. & x_{i+n+1}^{(n)} & -1 \end{vmatrix}$$

and, hence,

$$B_i^2=C_i^2$$

†N. A. Court; On Five Mutually Orthogonal Spheres, *Annals of Mathematics*, vol. 30 (1929), p. 613.

By means of 2) and 3), we have

$$A_iB_i = \begin{vmatrix} r_{i+1}^2+c_{i+1} & c_{i+2} & c_{i+3} & \cdots & c_{i+n+1} \\ c_{i+1} & r_{i+2}^2+c_{i+2} & c_{i+3} & \cdots & c_{i+n+1} \\ c_{i+1} & c_{i+2} & r_{i+3}^2+c_{i+3} & \cdots & c_{i+n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i+1} & c_{i+2} & c_{i+3} & \cdots & r_{i+n+1}^2+c_{i+n+1} \end{vmatrix}$$

To the first row of this determinant, add the other rows, then subtract the $(j+1)$st row from the jth row, $(j=2,3,\ldots,n)$ and subtract $(n-1)$ times the last row from the first row. Next, add the $(n-k)$th row to the $(n-k-1)$st row, $(k=0,1,2,\ldots,n-3)$; finally, multiply the first row by $-\frac{1}{2}$, the second, third,..., nth row by $\frac{1}{2}$ and add them to the last row. Then, making use of 6) and factoring out $\frac{1}{2}$ from the last row, we have

$$A_iB_i = \frac{1}{2}\begin{vmatrix} R_i^2 & R_i^2 & R_i^2 & \cdots & R_i^2 & R_i^2-(n-1)r_{i+1+n}^2 \\ 0 & r_{i+2}^2 & 0 & \cdots & 0 & -r_{i+n+1}^2 \\ 0 & 0 & r_{i+3}^2 & \cdots & 0 & -r_{i+n+1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & r_{i+n}^2 & -r_{i+n+1}^2 \\ -r_{i+1}^2 & 0 & 0 & \cdots & 0 & r_{i+n+1}^2 \end{vmatrix}$$

$$= \tfrac{1}{2}R_i^2 \prod_{j=1}^{n+1} r_{i+j}^2 \Big\{ \sum_{j=1}^{n+1} 1/r_{i+j}^2 - 1/r_i^2 \Big\}.$$

Hence,

$$2c_i = R_i^2+(n-1)r_i^2, \tag{8}$$

and

$$\sum x_i^2 = R_i^2+nr_i^2. \tag{9}$$

4. Theorems. From 9) it follows that the radius P_j of the hypersphere with center P_j and orthogonal to its associated central hypersphere is given by

$$P_j^2 = nr_j^2.$$

and

Theorem 1. The sum of the reciprocals of the squares of the radii of the hyperspheres whose centers are $n+2$ orthocentric points and which are orthogonal to the associated central hyperspheres of their centers vanishes.

Using 6), we obtain

Theorem 2. The sum of the squares of the radii of the hyperspheres circumscribing the associated central $(n+1)$-points of the centers of $n+2$ mutually orthogonal hyperspheres is equal to $(n^2-n+2)/4$ times the sum of the squares of the radii of the orthogonal hyperspheres.

Using 9), we obtain

Theorem 3. The sum of the squares of the distances of the centers of $n+2$ mutually orthogonal hyperspheres from the centers of their associated central hyperspheres is equal to $(n+1)(n+2)/4$ times the sum of the squares of the radii of the orthogonal hyperspheres.

The center of gravity G of the centers of $n+2$ mutually orthogonal hyperspheres is given by $G \equiv (n/(n+2)\cdot x_i)$ and the center of gravity G_i of the associated central $(n+1)$-point T_i is given by

$$G_i \equiv ((n-1)/(n+1)\cdot x_i).$$

Then,

Theorem 4. The center of gravity of $n+2$ orthocentric points lies on the join of the circumcenter and orthocenter (the Euler line) of each associated central $(n+1)$-point and divides the segment internally in the ratio $n:2$.

Theorem 5. The center of gravity of an associated central $(n+1)$-point lies on its Euler line and divides the segment internally in the ratio $n-1:2$.

Theorem 6. The sum of the squares of the distances of the centers of $n+2$ mutually orthogonal hyperspheres from the centers of gravity of their associated central $(n+1)$-points is equal to $(n+2)/(n+1)$ times the sum of the squares of the radii of the orthogonal hyperspheres.

Theorem 7. The sum of the squares of the distances of the centers of $n+2$ mutually orthogonal hyperspheres from the center of gravity of these centers is equal to $(n+1)/(n+2)$ times the sum of the squares of the radii of the orthogonal hyperspheres.

Theorem 8. The sum of the squares of the distances of the center of gravity of the centers of $n+2$ mutually orthogonal hyperspheres to the centers of gravity of the associated central $(n+1)$-points is equal to $1/(n+1)(n+2)$ times the sum of the squares of the radii of the orthogonal hyperspheres.

Theorem 9. The sum of the powers with respect to $n+2$ mutually orthogonal hyperspheres of the centers of their associated central

hyperspheres is $(n^2+3n-2)/4$ times the sum of the sum of the squares of the radii of the orthogonal hyperspheres.

Theorem 10. The sum of the powers of the center of gravity of an orthocentric $(n+1)$-point with respect to the $n+2$ mutually orthogonal hyperspheres associated with it is equal to $-1/(n+1)^2$ times the sum of the squares of the edges of the $(n+1)$-point.

Theorem 11. The sum of the powers with respect to $n+2$ mutually orthogonal hyperspheres of the centers of gravity of their centers is equal to $-1/(n+2)$ times the sum of the square of the radii of the orthogonal hyperspheres.

5. The first $n(n+1)$-point hyperspheres.

Theorem 12. Given $n+2$ mutually orthogonal hyperspheres, the midpoints of the edges of an associated central $(n+1)$-point T_i and the points P_{jk}, $(j,k\neq i)$, lie on a hypersphere, the first $n(n+1)$-point hypersphere, of T_i whose center T_i bisects the join of the circumcenter and orthocenter of T_i and the square of whose radius is

$$\frac{1}{16}\left\{(n-3)^2r_i^2+\sum_{j=1}^{n+2}r_{i+j}^2\right\}.$$

Theorem 13. The sum of the squares of the radii of the first $n(n+1)$-point hyperspheres associated with $n+2$ mutually orthogonal hyperspheres is equal to $(n^2-5n+10)/16$ times the sum of the squares of the radii of the orthogonal hyperspheres.

Theorem 14. The sum of the powers of the centers of $n+2$ mutually orthogonal hyperspheres with respect to the first $n(n+1)$-point hyperpsheres of their associated central $(n+1)$-points is equal to n times the sum of the squares of the radii of the orthogonal hyperspheres.

6. The second $n(n+1)$-point hyperspheres.

Consider an $(n+1)$-point T_i. The center of gravity of its face F_i opposite P_j is given by

$$\left(\frac{1}{n}\left\{\sum_{k=1}^{n+1}x_{i+k}^2-x_i\right\}\right)$$

and the orthocenter of F_{ij} (the intersection of P_iP_j and F_{ij}) is given by

$$\left(\frac{r_j^2x_i+r_i^2x_j}{r_i^2+r_j^2}\right)$$

Theorem 15. The centers of gravity and orthocenters of the faces and the points which divide the joins of the vertices and orthocenter of an orthocentric $(n+1)-$point internally in the ratio $n-1:1$ lie on a hypersphere, the second $n(n+1)-$point hypersphere, whose center divides the join of the circumcenter and orthocenter in the ratio $n-1:1$ and whose radius is $1/n$ times the radius of the circumscribing hypersphere of the $(n+1)-$point.

An altitude P_iP_j of T_i meets the second $n(n+1)-$point hypersphere again in the point ξ_{ij} given by

$$\xi_{ij} \equiv \left(\frac{1}{n}x_i + \frac{n-1}{n}x_j \right)$$

Theorem 16. Given an orthocentric $(n+1)-$point T_i, the $(n+1)-$point whose vertices are $\xi_{ij}, j \neq i$, is orthocentric. Its circumscribing hypersphere and orthocenter is the second $n(n+1)-$point hypersphere and orthocenter of T_i.

7. Other related hyperspheres.

Theorem 17. The points P_{jk} lying on the edges of an orthocentric $(n+1)-$point T_i, $(i \neq j,k)$ issuing from P_j lie on a hypersphere S_{ij} having P_iP_j as diameter.

Theorem 18. The hypersphere S_{ij} is orthogonal to S_k and to S_{k2}, $(i \neq j \neq k \neq l)$.

Theorem 19. The midpoints of the joins of the orthocenter of T_i and its vertices lie on a hypersphere μ_i of radius one-half that of the circumscribing hypersphere and concentric with the first $n(n+1)$ $-$point hypersphere of T_i.

When $O_i \equiv (0)$, then $O_j \equiv (n/2.\ (x_i - x_j))$

and $T_i \equiv (\frac{1}{4}\{nx_i - (n-2)x_j\})$.

Theorem 20. Given $n+2$ mutually orthogonal hyperspheres, the sum of the squares of the radii of the hyperspheres concentric with the circumscribing hyperspheres of T_j and orthogonal to $\mu_i, (j \neq i)$, is equal to $n(n-1)/8$ times the sum of the squares of the edges of T_i.

Theorem 21. Given $n+2$ mutually orthogonal hyperspheres, the sum of the powers of the centers of the associated central hyperspheres with respect to $\mu_k (k = 1,2,\ldots,n+2)$ is equal to $n^3/4$ times the sum of the squares of the radii of the orthogonal hyperspheres.

7. Related orthocentric systems.

Theorem 22. The circumcenters of the associated $(n+1)-$points of an orthocentric system in $n-$space form another orthocentric system homothetic to the given one in the ratio $-n:2$.

Theorem 23. The centers of gravity of the associated central $(n+1)-$points of an orthocentric system form another orthocentric system homothetic to the given one in the ratio $-1:n+1$.

Theorem 24. The centers of the first $n(n+1)-$point hyperspheres associated with an orthocentric system form another orthocentric homothetic to the given one in the ration $-(n-2):4$.

Theorem 25. The centers of the second $n(n+1)-$point hyperspheres associated with an orthocentric system form another orthocentric system homothetic to the given one in the ration $(n-2); 2n$.

Announcement

Perhaps the most significant questionnaire pertaining to mathematics that has ever been broadcast to American teachers of the subject, is one recently inaugurated by Professor Seidlin, editor of our Teacher's Department, and supervised by him from his office in Alfred University, under the auspices of the National Mathematics Magazine, and financed by Louisiana State University. Professor Seidlin's project is fundamentally concerned with: (a) questions relating to the quantity, place in the curriculum and organized form of mathematics in the secondary schools, (b) the age-old query, Is, or is not, mathematical ability a monopoly of the few?

A late communication from Professor Seidlin contains the following: "I am quite pleased with the returns thus far, particularly with the comments. Not only is the little space provided for comments filled, but in the majority instances the back page is utilized as well. In a good many cases one and two page letters accompany the questionnaire." ..... "My statistician friends warned me not to expect more than 350 replies. I have received 600 already, and they are still coming in at the rate of 25 a day."

In view of this landslide of responses to the questionnaire addressed to one thousand American teachers of mathematics, the readers of National Mathematics Magazine may confidently anticipate that the Teacher's Department of our May issue, in which Editor Seidlin's findings are to be reported, will be unusually stimulating and informing. Thus, it is important that present non-subscribers to National Mathematics Magazine who may wish to follow Professor Seidlin's analysis of returns, should take steps to acquire possession of his complete report by promptly sending in their subscription orders. New subscriptions coming in before our May issue will be allowed to include along with the May issue all issues of Volume 11.

S. T. SANDERS,
Editor and Manager.

An Approximate Formula for Evaluating Factorial Numbers

By GEORGE R. TOWN, D. E.
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ABSTRACT

It is shown that $\lim_{n=\infty} \left\{ [(n+1)!]^{1/n+1} - [n!]^{1/n} \right\} = 1/e$ where e is the base of the Naperian system of logarithms. From this is derived the formula $n! = (a+bn)^n$. Values of the parameters a and b are given which allow the determination of factorial n with an error of less than one per cent for all values of n up to 100,000.

Stirling's series for the natural logarithm of factorial n is*

$$\log n! = (n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log (2\pi) + \frac{B_1}{1 \cdot 2 \cdot n} - \frac{B_2}{3 \cdot 4 \cdot n^3} + \ldots\ldots \qquad (1)$$

where n is a positive integer and B_1, B_2, ... are Bernoulli's numbers.

From this series, Stirling's approximation for n!, which holds for large values of n, is easily obtained. It is

$$n! = \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \qquad (2)$$

where e is the base of the Naperian system of logarithms.

It is the purpose of this paper to derive another approximation for n!, to evaluate the constants in the new expression, and to indicate the limits of accuracy which can be obtained in its use. The new approximation does not yield as accurate results as does Stirling's, but it is somewhat more simple in its form and the computation of n! is considerably simplified.

From equation (1) it is evident that

$$\log (n!)^{1/n} = \frac{n+\frac{1}{2}}{n} \log n - 1$$

**An Introduction to the Theory of Infinite Series*; T. J. I'a. Bromwich; Macmillan and Co., London; 2d Edition, 1926; Article 108, pages 329-331, Articles 178-180, pages 506-513.

$$+\frac{1}{2n}\log(2\pi)+\frac{B_1}{1\cdot 2\cdot n^2}-\frac{B_2}{3\cdot 4\cdot n^4}+\ldots. \tag{3}$$

When n is large, this reduces to

$$\log(n!)^{1/n}=\log n-1+\frac{1}{2n}\log(2\pi n) \tag{4}$$

Hence,

$$(n!)^{1/n}=n\,(2\pi n)^{1/2n}e^{-1} \tag{5}$$

Again,

$$[(n+1)!]^{1/n+1}=(n+1)\,[2\pi(n+1)]^{1/2(n+1)}e^{-1} \tag{6}$$

Subtracting (5) from (6),

$$[(n+1)!]^{1/n+1}-[n!]^{1/n}=\left\{(n+1)\,[2\pi(n+1)]^{1/2(n+1)}-n(2\pi n)^{1/2n}\right\}e^{-1} \tag{7}$$

When n is large,

$$[2\pi(n+1)]^{1/2(n+1)}\doteq(2\pi n)^{1/2n} \tag{8}$$

Substituting (8) in (7)

$$[(n+1)!]^{1/n+1}-[n!]^{1/n}=(2\pi n)^{1/2n}e^{-1} \tag{9}$$

Now, when n is sufficiently large,

$$(2\pi n)^{1/2n}\doteq 1 \tag{10}$$

Even for smaller values of n, the factor $(2\pi n)^{1/2n}$ is practically constant over fairly wide ranges in n. As a first order approximation, then, it can be said that

$$[(n+1)!]^{1/n+1}-[n!]^{1/n}=d \tag{11}$$

where

$$d=(2\pi n)^{1/2n}e^{-1} \tag{12}$$

and

$$\lim_{n=\infty} d=\frac{1}{e}=0.36787945 \tag{13}$$

It should be noted that

$$\lim_{n=\infty}(n!)^{1/n}=\infty^{\circ}$$

Frequently, the form ∞° is equal to unity. In this case, however, ∞° is not unity for otherwise $[(n+1)!]^{1/n+1}-[n!]^{1/n}$ would approach zero as n increases indefinitely.

From equation (11) it is evident that

$$n! = \left\{[(n-1)!]^{1/n-1}+d\right\}^n \tag{14}$$

If the value

$$[(n_0-1)!]^{1/n_0-1}=c \tag{15}$$

is known and it is desired to compute n! where n is not too greatly different from n_0, equation (14) may be rewritten as

$$n! = [c+(n-n_0)b]^n \tag{16}$$

where b is the average value of d in the interval between n and n_0.

Simplifying equation (16), we have

$$n! = (a+bn)^n \tag{17}$$

where

$$a = c-bn_0 \tag{18}$$

It should be noted that when n is very great, equation (17) and Stirling's formula, equation (2), approach the same value.

By Stirling's formula,

$$\lim_{n=\infty} n! = \left(\frac{n}{e}\right)^n \tag{19}$$

By equation (17),

$$\lim_{n=\infty} n! = (bn)^n = \left(\frac{n}{e}\right)^n \tag{20}$$

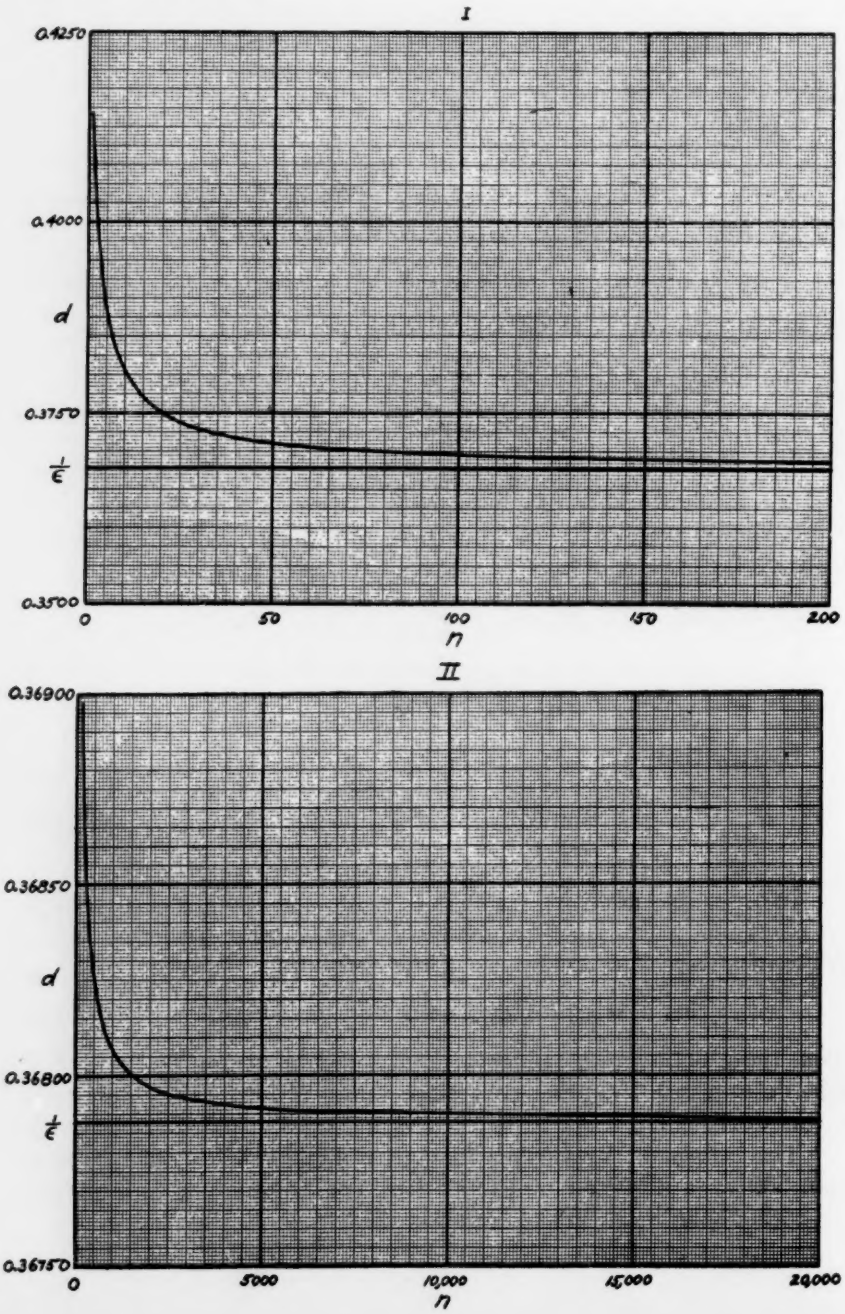
In order that equation (17) can be used, it is necessary to evaluate the constants a and b. The values of $\log_{10} n!$ were computed for all values of n from 1 to 500 by use of the expression

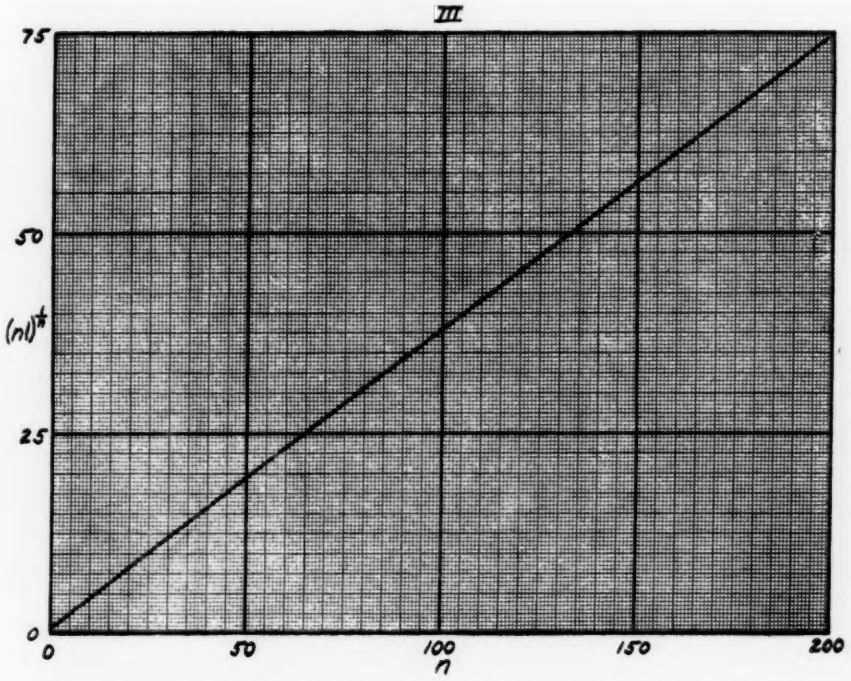
$$\log n! = \log 1+\log 2+\log 3+\ldots.+\log n \tag{21}$$

From these figures, the values of $\log_{10}[(n!)^{1/n}]$, of $(n!)^{1/n}$ and of d were determined. Vega's seven place tables of logarithms were used in this work.

The values of the various functions of n for values of n greater than 500 were obtained by use of Stirling's approximation, equation (2), rather than by means of equation (21).

Values of d as a function of n for two ranges of n are plotted in Figs. 1 and 2. Fig. 3 shows the curve of $(n!)^{1/n}$ as a function of n for values of n up to 200.





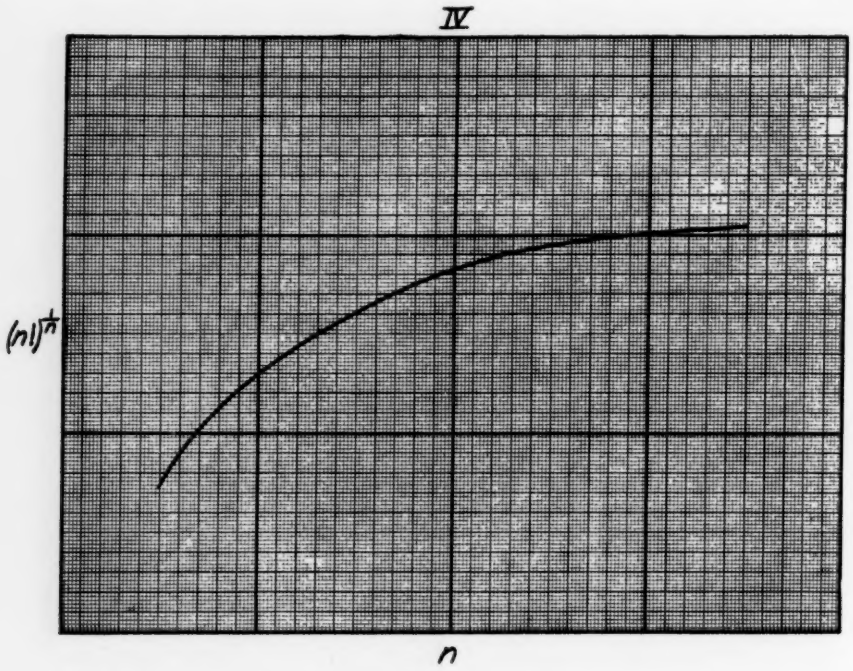
The most precise procedure to obtain the co-efficients a and b from these data would be to obtain the mean linear curve of $(n!)^{1/n}$ as a function of n for various ranges of n by least square methods. However, it was felt that this amount of labor was not justified as the results are only approximate at the best. Hence, a less tedious method of evaluation was used.

In Fig. 4 a plot of $(n!)^{1/n}$ versus n is shown with the curvature greatly exaggerated. For values of n less than 500, the mean straight line throughout the range in n from n_1 to n_2 was assumed to be that which intersected the actual curve at the points

$$\left[n_1 + \frac{n_2 - n_1}{4}\right] \text{and} \left[n_1 + \frac{3(n_2 - n_1)}{4}\right].$$

The value of n_0, equation (15), was taken as

$$n_0 = n_1 + \frac{n_2 - n_1}{4} \tag{22}$$



The average value of d over the range n_1 to n_2 or of b was

$$b = \frac{1}{n_2 - n_1}[(n_2!)^{1/n_2} - (n_1!)^{1/n_1}] \tag{23}$$

From these values, the constant a was obtained by means of equations (18) and (15).

These values of a were not the final ones chosen as their use resulted in excessive errors at the two ends of the range, near n_1 and n_2, and very low errors in the middle of the range. In order to reduce the errors at the ends of the range, the value of a was changed until these errors were equal, but opposite in sign, to those at the middle. This increased these latter errors, but reduced the maximum errors.

For values of n greater than 500, n_0 was taken as n_1. The value of b was again computed by means of equation (23). The resulting value of a yielded correct values of n! at the ends of the range but low values at the middle. The value of a was then adjusted as before to reduce the maximum errors.

The final values of a and b which are to be used in equation (17) for various ranges in value of n are given in the table. This table

also gives the maximum errors which will be obtained when these constants are used. It is seen that the errors in factorial n are always less than one per cent. It is obvious that if the range in values of n were more restricted, the errors would be reduced.

The advantage of equation (17) over Stirling's approximation, equation (2), is that the labor of numerical computation is considerably reduced, especially if the factorial of several successive numbers is to be computed. This may be best illustrated by a numerical example. Let it be required to compute 160! and 161!

The true value of log 160! is 284.6734564.

By Stirling's formula,

$$160! = \left(\frac{160}{e}\right)^{160} \sqrt{2\pi^{\times 160}}$$

$$\begin{aligned}
\log 160 &= 2.2041200 \\
\log\ e &= \textit{0.4342945} \\
&\quad 1.7698255 \\
\log 1.7698255 &= 0.2479304 \\
\log\ 160 &= \textit{2.2041200} \\
\log\ 283.172 &= 2.4520504 \\
\log\ 2 &= 0.3010300 \\
\log\ \pi &= \textit{0.4971499} \\
&\quad 0.7981799 \\
\log 160 &= \textit{2.2041200} \\
&\quad 2\,|\,\textit{3.0022999} \\
&\quad 1.5011500 \\
&\quad \textit{283.172} \\
&\quad 284.673 = \log 160!
\end{aligned}$$

By equation (17) and Table I,

$$160! = (1.12524 + 160 \times 0.36888)^{160}$$

$$\begin{aligned}
\log\ 0.36888 &= 9.5668851 \\
\log\ 160 &= \textit{2.2041200} \\
\log\ 59.0208 &= 1.7710051 \\
\textit{1.12524} & \\
\log\ 60.1460 &= 1.7792067 \\
\log 1.7792067 &= 0.2502264 \\
\log\ 160 &= \textit{2.2041200} \\
\log\ 284.673 &= 2.4543464 \\
\log 160! &= 284.673
\end{aligned}$$

The true value of log 161! is 286.8802823.

By Stirling's formula,

$$\begin{aligned}
\log 161 &= 2.2068259 \\
\log\ e &= \textit{0.4342945} \\
&\quad 1.7725314 \\
\log 1.7725314 &= 0.2485939 \\
\log\ 161 &= \textit{2.2068259} \\
\log\ 285.378 &= 2.4554198 \\
\log 2\pi &= 0.7981799 \\
\log 161 &= \textit{2.2068259} \\
&\quad 2\,|\,\textit{3.0050058} \\
&\quad\quad 1.5025029 \\
&\quad \textit{285.378} \\
&\quad 286.881 = \log 161!
\end{aligned}$$

By equation (17),

$$\begin{aligned}
a+160b &= 60.1460 \\
b &= \textit{0.3689} \\
\log\ 60.5149 &= 1.7818623 \\
\log 1.7818623 &= 0.2508741 \\
\log\ 161 &= \textit{2.2068259} \\
\log\ 286.880 &= 2.4577000 \\
\log 161! &= 286.880
\end{aligned}$$

It is seen that to compute log 160! by Stirling's formula, 14 lines of work are required. If equation (17) is used, 8 lines are needed. To then compute log 161! by Stirling's formula requires 12 lines of work while if equation (17) is used, only 6 lines are necessary.

SUMMARY

1. It has been shown that an approximate formula for factorial n is

$$n! = (a+bn)^n$$

2. Values of the parameters a and b for several ranges in value of n between 10 and 100,000 are given. When the correct parameters are used, the resulting value of n! will not be in error by more than one per cent.

3. While this formula does not give as accurate results as Stirling's approximation, it is somewhat easier to use and it at least gives another analytical expression for factorials.

TABLE I

Values of the Constants a and b

n	*a*	*b*	*Maximum Per cent Error in* n!
10-20	0.75652	0.37756	0.82
20-35	0.83562	0.37357	0.65
35-60	0.91299	0.37136	0.70
60-100	0.99325	0.37002	0.68
100-150	1.06742	0.36927	0.46
150-200	1.12524	0.36888	0.24
200-300	1.18443	0.36859	0.53
300-500	1.25550	0.36835	0.96
500-750	1.3438	0.36817	0.51
750-1000	1.4025	0.36809	0.35
1000-1500	1.4621	0.36803	0.75
1500-2000	1.5366	0.36798	0.42
2000-3000	1.6001	0.36795	0.81
3000-4000	1.6621	0.36793	0.35
4000-5000	1.7030	0.36792	0.24
5000-6000	1.7546	0.36791	0.54
6000-8000	1.7872	0.367905	0.27
8000-10,000	1.8275	0.367900	0.19
10-15,000	1.8890	0.367894	0.6
15-20,000	1.9490	0.367890	0.3
20-30,000	2.0090	0.367887	0.3
30-50,000	2.1010	0.367884	0.8
50-75,000	2.1830	0.3678824	0.8
75-100,000	2.2485	0.3678815	0.3
100,000 up	2.395	0.3678800	(0.8)

The Teacher's Department

Edited by
JOSEPH SEIDLIN

Introduction

Teaching the Concept of Directional Derivative

By EDWIN G. OLDS
Carnegie Institute of Technology

When Professor Seidlin approached me at the St. Louis meeting with the request that I write a report on my method of teaching some topic of elementary collegiate mathematics, I expressed to him my conviction that anything I might offer would be neither novel nor flawless. After two decades of teaching and considerable experience in trying to teach teachers to teach, I must confess little confidence in my ability to impart to any given class the major portion of the knowledge which I think they should acquire in any stated period. However, if the plan of attack which I shall discuss in this paper should prove of any benefit to younger teachers, I am glad to add it to the all too scant literature on the teaching of collegiate mathematics.

The presentation of any topic to a class should contain three constituents: review, view, and preview. To explain my meaning more fully: in the presentation of new material, every opportunity should be grasped to review principles previously studied, then the new principles should be carefully examined, and finally, paths for future exploration should be indicated. In this paper, I have tried to show how to combine review, view, and preview in presenting to a class the concept of directional derivative.

THE LESSON

When z is a function of x and y, we have found that the increment of z may be expressed in terms of partial derivatives and infinitesimals. Let us consider that relation. We have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are infinitesimals which approach zero with Δx and Δy. In our last lesson we noticed that if x and y were functions of t

we could divide Δz by Δt and take the limit as Δt approached zero. We found that

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

Is this expression for dz/dt exact or approximate?
What do you say, Jackson?
(Jackson says it is exact and explains why.)
Now suppose we consider

$$z=4-(x^2+4y^2)$$

I'll sketch it in rapidly. (See Fig. 1.) When x and y are zero, z is 4, when $z=0$, we have the ellipse,

$$x^2+4y^2=4.$$

In the XY plane we have an ellipse and in the XZ and YZ planes we have parabolas. (The curves are drawn tangent to the guide lines, See Fig. 1). (The class is familiar with the fact that the surface also

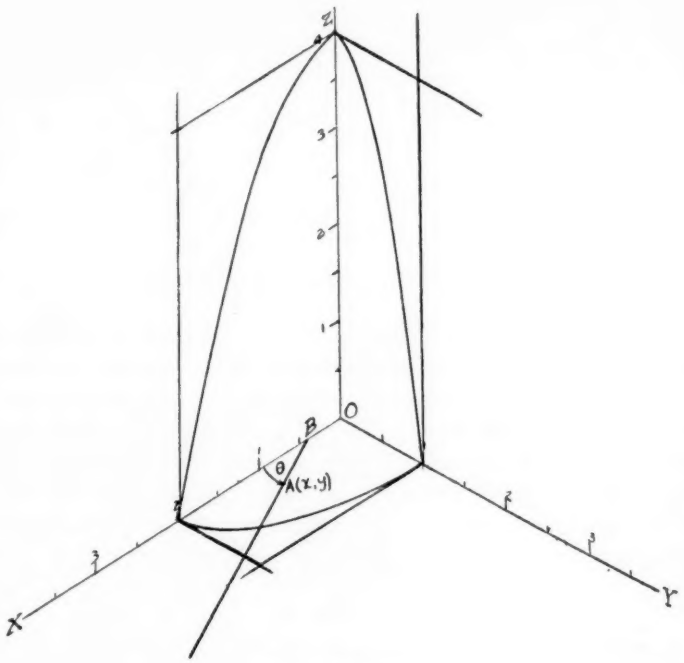


Fig. 1

lies in other octants). Now consider a point A in the XY plane. Let's draw a line thru A, meeting OX in B. Suppose that the distance of A measured along the line from some reference point such as B, is r and call the angle XBA, θ. Now x and y will depend on r and θ.

Perhaps we might look at it in two dimensions. (See Fig. 2.)

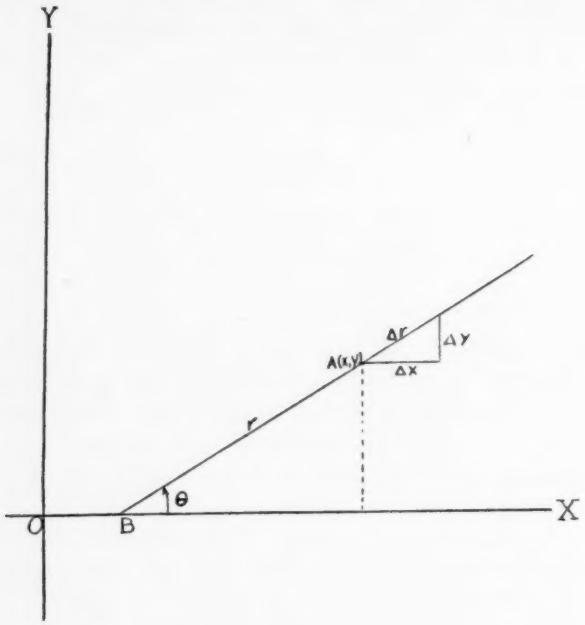


Fig. 2

$$y = r\sin\theta,\ x = OB + \gamma\cos\theta$$

If we move along the line, then since θ is constant x and y depend on r only. Corresponding to an increment of r, Δr we get an increment of x, Δx and an increment of y, Δy. Now let us see what happens in three dimensions. (See Fig. 3). Pass a plane thru AB perpendicular to XY. This cuts the surface in a parabola. Corresponding to A is a point P on the surface, with co-ordinates x, y, z. Now if I change r an amount Δr, I not only change x an amount Δx and y an amount Δy but also change z an amount Δz and

$$CQ = z + \Delta z$$

where in this case, Δz is negative. Now consider $\Delta z/\Delta r$. This is the tangent of the angle which the secant PQ makes with a horizontal

line through P in the plane PQAC. As Δr approaches zero, it is clear that $\Delta z/\Delta r$ approaches dz/dr since z depends on r and the secant PQ approaches the tangent at P. So, geometrically, dz/dr gives the slope of the tangent at P. From another point of view, dz/dr gives the rate of change of z with respect to r at P in the direction in which r is measured. We call dz/dr the "directional derivative."

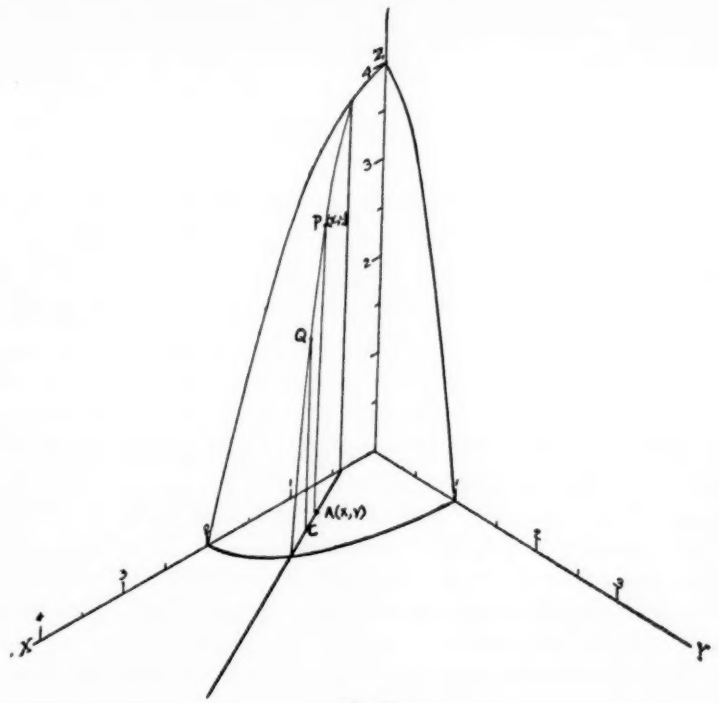


Fig. 3

Returning now to our expression for Δz, let's divide by Δr. Dividing and taking the limit as Δr approaches zero, we have

$$\lim_{\Delta r\to 0}\frac{\Delta z}{\Delta r}=\frac{\partial z}{\partial x}\lim_{\Delta r\to 0}\frac{\Delta x}{\Delta r}+\frac{\partial z}{\partial y}\lim_{\Delta r\to 0}\frac{\Delta y}{\Delta r}$$

$$+\epsilon_1\lim_{\Delta r\to 0}\frac{\Delta x}{\Delta r}+\epsilon_2\lim_{\Delta r\to 0}\frac{\Delta y}{\Delta r}$$

or

$$\frac{dz}{dr}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dr}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dr}+0.$$

Why do ϵ_1 and ϵ_2 approach zero?

(After the proper answer is obtained, we continue the discussion.)

Now, let's examine dx/dr and dy/dr. Turning to our original expressions for x and y we see that since θ is fixed,

$$\frac{dx}{dr} = \cos\theta \text{ and } \frac{dy}{dr} = \sin\theta$$

Then we can write

$$\frac{dz}{dr} = \frac{\partial x}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

Notice again that θ is the angle between PQAC and the XZ plane.

Suppose θ is 90°. Then we have $\cos\theta = 0$ and $\sin\theta = 1$ and $dz/dr = \partial z/\partial y$, which is what we might expect, since we have a plane parallel to ZY when we set $\theta = 90°$. Also set $\theta = 0°$ and we have

$$\frac{dz}{dr} = \frac{\partial z}{\partial x}.$$

So our partial derivatives may be considered special cases of the directional derivative.

Now let us imagine that our surface represents a mountain, standing on a plain. Suppose we are located at P. If we move in some directions from P, we will go down hill, in other directions we will go up hill. The rate at which we ascend or descend will depend on the direction in which we start.

In which direction should we start so as to stay on the same level? Suppose we set dz/dr equal to zero. First, however, let's compute dz/dr for this particular surface.

$$z = 4 - (x^2 + 4y^2)$$

$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -8y$$

$$\frac{dz}{dr} = -2x\cos\theta - 8y\sin\theta$$

If we set dz/dr equal to 0, then

$$-2x\cos\theta - 8y\sin\theta = 0$$

or

$$-2x - 8y\tan\theta = 0$$

But, if $\tan\theta$ is the slope of a tangent line we can replace it by dy/dx. Then

$$-2x - 8y\frac{dy}{dx} = 0$$

What would we differentiate to get this?
Wouldn't it come from

$$x^2 + 4y^2 = C?$$

What we have done is to solve a very simple type of differential equation. A little later you will become better acquainted with differential equations but this one is so easy that it seems nice to solve it now. Notice that the solution tells us that the curve whose slope is related to its abscissa and ordinate in the way indicated by our first equation is the ellipse, $x^2 + 4y^2 = C$.

Suppose we draw this ellipse through A (See Fig. 4). Then, if we choose a direction tangent to the curve through A, we have a direction from P such that $dz/dr = 0$. The ellipse through A is the projection of a curve through P. If we follow the curve through P, we will neither ascend nor descend the mountain. What do we call the curve thru P? (Some of my embryo Civil Engineers answer, "Contour line.")

Now, let's think about another problem. In what direction should we start from P so that we will ascend the fastest? In other words, so that the rate of increase of height with respect to horizontal distance is the greatest.

$$\frac{dz}{dr} = -2x\cos\theta - 8y\sin\theta$$

For x and y fixed, dz/dr depends on θ. Then our question resolves itself into searching for the value of θ which will make dz/dr a maximem. Differentiating with respect to θ,

$$\frac{d}{d\theta}\left[\frac{dz}{dr}\right] = 2x\sin\theta - 8y\cos\theta.$$

$$2x\sin\theta - 8y\cos\theta = 0$$

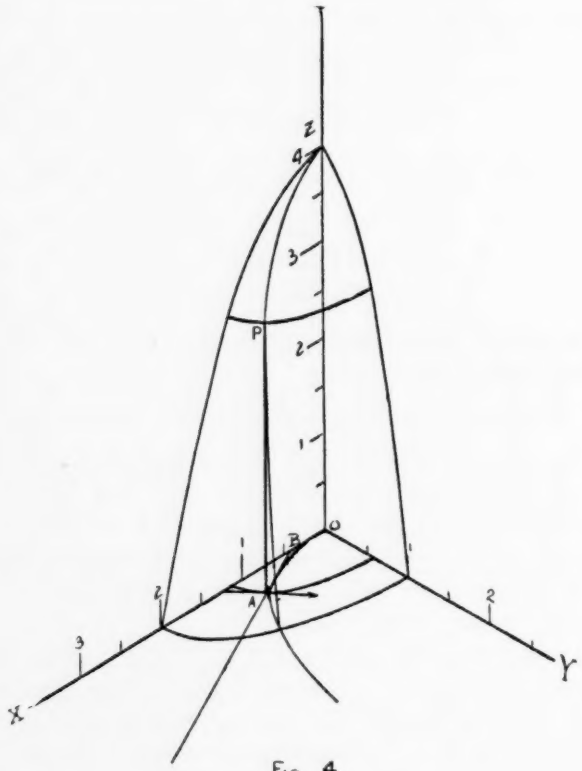


Fig 4

for $\tan\theta = 4y/x$

If $4y/x$ is positive, θ may lie in quadrants one or three. Taking the second derivative,

$$\frac{d^2}{d\theta^2}\left(\frac{dz}{dr}\right) = 2x\cos\theta + 8y\sin\theta$$

For θ in quadrant one, the second derivative is positive and dz/dr is a minimum; for θ in quadrant three, dz/dr is a maximum. Noting that the two angles differ by 180°, it is apparent that we travel in the same plane but in opposite directions. (At this point, I illustrate on the figure, just what I mean.)

Let's return to the equation

$$2x\sin\theta - 8y\cos\theta = 0,$$

and as before, replace tanθ by dy/dx. Then

$$2x\frac{dy}{dx} - 8y = 0.$$

This equation results from differentiating $y/x^4 = C$ as you can readily verify. You can also verify that this curve is perpendicular to our ellipse. Then thru A, we have two curves which, as we say, are orthogonal. The curve through A is the projection of a curve through P which would be the steepest path through P. This is sometimes called a gradient line.

Although a detailed investigation of these interesting curves must be postponed until later, some of you may be interested in sketching the pairs of curves for different positions of P.

Today we have been mainly concerned with one particular application of the directional derivative. With other interpretations for z we could interpret our pairs of lines as isothermal lines and lines of flow, or equipotential lines and lines of force. As you think about the matter, other applications will, no doubt, occur to you.

AFTERWORD

It has been my endeavor to reproduce, as faithfully as I could, the discussion which took place in my class when we first considered the Directional Derivative. Comments and questions of students have largely been omitted, in order to avoid breaking the continuity too often. My class is composed of some of the better engineering sophomores. The discussion given above consumed about an hour. At subsequent meetings of the class much of the discussion was repeated and the students gained better understanding of the principles by solving a number of thought provoking problems.

Mathematical Notes

Edited by
L. J. ADAMS and I. MAIZLISH

The fall meeting of the Maryland-District of Columbia-Virginia section of the M. A. A. was held at the University of Maryland, College Park, Maryland on Saturday, December 7th, 1935. Professor G. T. Whyburn of the University of Virginia presided over the meetings. Papers were read by R. T. Zoch of the United States Weather Bureau, Dr. E. W. Woolard of the United States Weather Bureau, Dr. A. Sinkov of Washington, Professor W. F. Shenton of American University, Professor H. M. Robert of the United States Naval Academy, Professor R. E. Gaines of the University of Richmond, and Professor E. J. McShane of the University of Virginia. The spring meeting is scheduled for May 9th and will be reported in the next issue of this magazine.

The fourth annual meeting of the Wisconsin section of the M. A. A. will be held on Saturday, May 9, 1936 at the University of Wisconsin. The program includes the following papers:

1. *Extension of the Concept of Length.* Professor R. L. Jeffery of Acadia University, Canada.
2. *Some Topics on the Teaching of Mathematics.* Professor M. L. Hartung of the Wisconsin High School.
3. *A Topic on Linear Algebras.* Professor H. H. Conwell of Beloit College.
4. *The Moving $n-hedral$.* Professor G. A. Parkinson of the Milwaukee Extension Division of the University of Wisconsin.
5. *Graphical Representations of the Real and Complex Roots of Cubic Equations.* Mr. J. A. Ward, Assistant in Mathematics, University of Wisconsin.

Dr. R. A. Fisher, Galton Professor of Genetics of the University of London, is coming as Visiting Professor of Mathematics at Iowa State College, Ames, Iowa, for the first term of the Summer Session June 16-July 22, 1936. He will lecture on statistics and its applications in research.

The Iowa section of the M. A. A. will hold its twenty-fifth regular meeting on April 3-4, 1936, at Iowa City in conjunction with the fiftieth regular meeting of the Iowa Academy of Science.

The three hundred and twenty-ninth meeting of the American Mathematical Society was held at Columbia University, New York City on Saturday, February 29, 1936. Some fifteen papers were read by members of the Society. Eighteen others being presented by title.

Part I of E. H. Moore's book on *General Analysis* has just been published by the American Philosophical Society. It is devoted to the Algebra of Matrices in a form especially adapted to General Analysis.

Osiris is the name of the companion volumes of *Isis*, official journal of the History of Science Society. The volumes of *Osiris* are supplementary to *Isis*. They contain series of articles devoted to a single subject, and also longer memoirs. Both are excellent aids to mathematical historians.

The sixteenth regular meeting of the Southern California section of the M. A. A. was held on Saturday, March 7, 1936 at the Fullerton Junior College, Fullerton, California. The program included the following papers:

1. *Characterisitics of Mathematicians.* Professor E. T. Bell, California Institute of Technology.

2. *Elements of Non-Analytic Functions.* Professor E. R. Hedrick, University of California at Los Angeles.

3. *The Role of Objective in Teaching Junior College Mathematics.* Professor F. C. Touton, University of Southern California.

4. *The Solution of Linear Equations.* Mr. I. E. Highberg, California Institute of Technology.

5. *Elementary Congruences.* Professor C. G. Jaeger, Pomona College.

6. *Elements of a General Theory of Analytic Functions.* Mr. A. E. Taylor, California Institute of Technology.

7. *The Mathematics of the Torsion Balance.* Professor S. E. Urner, Los Angeles Junior College.

Americans will be interested in the entrance requirements of the English universities. The regulations for inspection and examination of schools of the University of London is a fair example. Their syllabus for mathematics includes the general headings algebra, graphic algebra and co-ordinate geometry, pure geometry, trigonometry and the introduction to differential calculus. The topics under these main divisions suggest our freshman college courses, and indicate the greater selectivity of the English school system. Back numbers of the examination sets can be obtained at a nomial cost from the universities. An

elementary reading knowledge of French or German is required of candidates for matriculation from the higher schools.

The Bulletin of the American Mathematical Society published as Part I of the January, 1936 issue, their Annual List of Published Papers. This list comprises all research papers read before the Society and subsequently published, including reference to the places of publication.

Beginning April 1st Professor A. F. Carpenter will be Executive Officer for the Mathematics Department at the University of Washington (Seattle).

In the spring quarter at the University of Washington Professor A. F. Carpenter and Professor J. P. Ballantine will try out their new text-book on Mathematics for Freshmen Engineers.

A new magazine devoted to the elucidation of concepts of elementary mathematics has appeared in Mexico. The new publication is called *Pitagoras*, is edited by Jorge Quijano and has its offices at Liverpool 30, Ciudad de Mexico. The first of the monthly issues is dated September, 1935.

L. J. ADAMS.

Problem Department

Edited by
T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

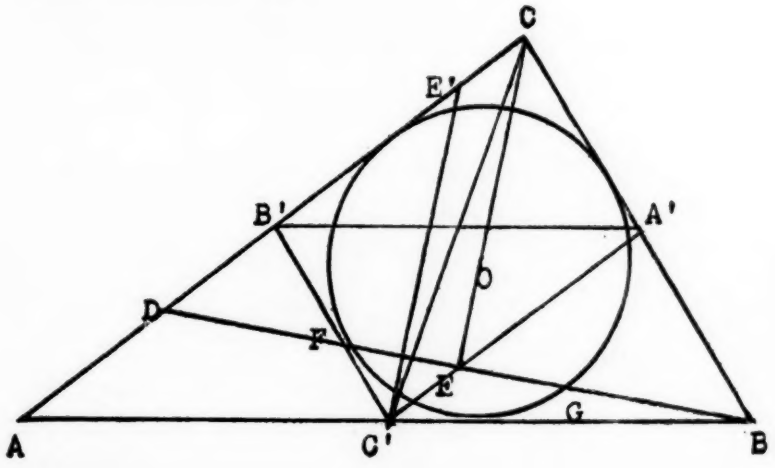
Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

SOLUTIONS

No. 110. Proposed by Walter B. Clarke.

The angles of a triangle are $A < B < C$ and C' is the midpoint of AB. A cevian from B to AC at D has its segments outside. The incircle equal. Show that the bisector of the angle at C' of medial triangle of $\triangle ABC$ is perpendicular to BD.

Solved by *A. C. Briggs.*



Let ABC be the given triangle. O the centre of the incircle. Lay off CD = side BC and draw BD. Through C′, the mid-point of AB, draw A′C′ parallel to AC and B′C′ parallel to BC. The triangle BCD is isosceles. A′C′B′C is a parallelogram and A′B′C′ is the medial triangle.

E is the mid-point of the base BD of the isosceles triangle BCD, and CE bisects the angle ACB, the chord FG, and passes through O, the centre of the incircle, and is perpendicular to BD. CE and C′E′ are bisectors of opposite angles of a parallelogram, and are therefore parallel. Hence, since CE is perpendicular to BD, C′E′ is likewise.

Q. E. D.

Also solved by *C. A. Balof* and *R. A. Miller.*

No. 111. Proposed by Walter B. Clarke.

O and O′ are centers of two unequal circles intersecting each other at S and T. Line OO′ cuts the circles at D,F,G, and E in that sequence. Show that

$$\sqrt{\overline{DF}\cdot\overline{FG}\cdot\overline{GE}\cdot\overline{DE}}=\overline{OO'}\cdot\overline{ST}$$

Solution by *C. A. Balof.*

Let r and R be the radii of circles with centers O and O′ respectively. Let ST and OO′ intersect at K. Then $DF\cdot DE\cdot FG\cdot GE$

$$=(\overline{O'D}^2-R^2)(R^2-\overline{O'G}^2)$$

$$=(\overline{OO'}^2+2r\cdot OO'+r^2-R^2)(R^2-\overline{OO'}^2+2r\cdot OO'-r^2)$$

$$=(\overline{OO'}^2+2r\cdot OO'+\overline{OK}^2-\overline{O'K}^2)(\overline{O'K}^2-\overline{OK}^2-\overline{OO'}^2+2r\cdot OO')$$

$$=\overline{OO'}^2(OO'+2r+OK-O'K)(O'K-OK-OO'+2r)$$

$$=4\,\overline{OO'}^2(r^2-\overline{OK}^2)$$

$$=4\,\overline{OO'}^2\cdot\overline{SK}^2$$

$$=\overline{OO'}^2\cdot\overline{ST}^2$$

Also solved by *A. C. Briggs, W. C. Janes, Wm. Benson Temple.*

PROBLEMS FOR SOLUTION

No. 121. Proposed by Walter B. Clarke.

Show that any verbisector (cevian from any vertex to the point halfway around the perimeter of a triangle), the incircle and its diameter perpendicular to the side cut by the verbisector are concurrent.

No. 122. Proposed by R. A. Miller.

Given the circle O with diameters AB and CD perpendicular. $\overset{\frown}{AQ}=\overset{\frown}{AS}=\overset{\frown}{CM}=\overset{\frown}{CR}=60°$ $AT=CV=\frac{2}{3}$ AO. Draw O′ through MVR, and O″ through QTS. Let P be the point of intersection of O′ and O″ which lies in O. Prove or disprove that $PAO=30°$.

No. 123. Proposed by R. A. Miller.

Determine the relations which exist between the angles of a triangle such that

(1) $a^2=b(b+c)$

(2) $bc^2=(a-b)^2(a+b)$

(3) $a^2bc^3=[(a^2-b^2)^2-b^2c^2]\,[(a^2-b^2)^2-bc]$

Where the sides are a, b, and c.

LATE SOLUTIONS

No. 106 by R. A. Miller.

No. 108 by R. A. Miller and A. C. Briggs.

Book Reviews

Edited by
P. K. SMITH

Plane Trigonometry. By Rietz, Reilly, and Woods. Macmillan Co., New York, May 1935. xx plus 142 pages, plus 72 pages of tables.

This very carefully prepared book differs in a number of particulars from the usual trigonometry text. The departures from the customary procedure, however, appear to have been adopted for sound or at least plausible pedagogical reasons rather than for the sake of variety. The keynote of the essential plan is explained in one sentence of the preface: "From teaching experience each of the authors feels that many students fail to grasp the essentials of trigonometry because too many new ideas are introduced at one time." In keeping with this policy even the six definitions of acute angle functions are broken up into two groups of three each, with enough exercises to aid the student in fixing the first group of definitions before considering the next.

In spite of the unhurried method, there is a rather surprising efficiency of results. For instance, the restricted definitions, the functions of 30°, 45°, and 60°, and the fundamental relations, together with numerous problems and some elementary identities, are disposed of in the first chapter of 13 pages. The whole conventional field for an elementary text, including an adequate treatment of logarithms, is covered in 142 pages.

The solution of the right triangle is introduced early, as usual, but the laborious computations without logarithms are avoided through the use of numbers of few digits and a three-place table of functions.

The book is well printed and attractive in makeup. The reviewer regrets his failure to find a sufficient number of flaws to establish his own expertness. As a feeble effort to this end, he suggests that the "Second Method of Reduction to Acute Angles" (the geometric method dealing with related angles) should really be the "First Method" in a book which emphasizes, as this one does, the practical teaching aspect of the subject. In any case, quibbling aside, it should be a pleasure to teach this text.

R. S. UNDERWOOD,
Texas Technological College.

The Status of Teachers of Mathematics in Secondary Schools of the United States. By Ben A. Sueltz, Courtland, New York, 1934.

The study under consideration was made in conjunction with work of the American Committee of the International Commission on Teaching of Mathematics, of which the author was a member. The report includes six chapters with Appendix and Bibliography. The scope includes Grammar School, Junior High School, Senior High School, and Junior College. We find forty tables, seven graphic diagrams and a summary at the end of each chapter, beginning with Chapter Three. It is impossible to give a summary of the report due to the fact that it is itself a summary of present practice. The tables are based largely on samples purported to reveal trends in such important matters as, years in which subject is taught, Degrees, credits in Mathematics and Education, Distribution of teachers by different methods, Age, Experience, Salary, Number of subjects in teacher schedule, Time distribution of training, etc. The tabulations do not include, length of tenure, grounds for appointment, variations in administrative practice, reasons for resignation or discharge, and items which school boards count in judging success or failure. It seems to the reviewer that such facts would be highly significant as regards status. A discrepancy appears in Table 3, page 16, where percentages are reported as 7, 93, and 9. The average salary reported seems to be high for certain sections of the country. Table 11 shows that 93 per cent receive $1000 or more per year. This may be due to the fact that too many items were selected from the eastern section where relative population is greater and school practice is better developed. Other data reported, such as special preparation, might also be considered in this connection. Perhaps tabulations on city systems, town systems, and rural systems in broad sections of fairly uniform population and wealth would give a better picture of the situation. I have in mind the Statistical Method of Lexis. In support of this comment, one state might be mentioned where contracts of approximately one-third of the teachers range well below $500.

The summaries and the conclusions of Chapter Six are especially interesting. Since we have not space in which to quote them, let us observe that the author has presented a striking array of facts and comments. One is startled by the great variety of ideas and of variation in current practice in all phases of the school situation. Variation among states would color a large map and the variety of ideas regarding contributions which subjects should make to so-called practical ends is bizarre indeed. When we read a study of such scope and com-

pleteness as this report, we wonder if the term School system is quite the term to use after all.

If we say railway system or chain store system, one understands immediately that objectives and policies emanate from a head office, the best available talent is employed and promoted through a long life of professional service, that values increase through service, and the philosophy of the system is that ultimate successes come in no other way. But when we say school system in America, just what do we mean? This study of Dr. Sueltz opens the way for more constructive thought. A companion study of Status of Administration, and one on Potential Contributions from Schools to Society, would bring closer to view the need for an American School System. Such studies should find place in every library and on the desk of every statesman.

C. D. SMITH,
Mississippi State College.